Remarks on the Liouville type problem in the stationary 3D Navier-Stokes equations

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Abstract

We study the Liouville type problem for the stationary 3D Navier-Stokes equations on $\mathbb{R}^3$. Specifically, we prove that if v is a smooth solution to (NS) satisfying $\omega = \operatorname{curl} v \in L^q(\mathbb{R}^3)$ for some $\frac{3}{2} \le q < 3$, and $|v(x)| \to 0$ as $|x| \to +\infty$, then either $v = 0$ on $\mathbb{R}^3$, or $\int_{\mathbb{R}^6} \Phi_+ dxdy = \int_{\mathbb{R}^6} \Phi_- dxdy = +\infty$, where $\Phi(x,y) := \frac{1}{4\pi} \frac{\omega(x)\cdot(x-y)\times(v(y)\times\omega(y))}{|x-y|^3}$, and $\Phi_\pm := \max\{0, \pm\Phi\}$. The proof uses crucially the structure of nonlinear term of the equations.

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1 Introduction

We consider the following stationary Navier-Stokes equations(NS) on $\mathbb{R}^3$.

$$(v \cdot \nabla)v = -\nabla p + \Delta v, \tag{1.1}$$
$$\operatorname{div} v = 0, \tag{1.2}$$

where $v(x) = (v_1(x), v_2(x), v_3(x))$ and $p = p(x)$ for all $x \in \mathbb{R}^3$. The system is equipped with the boundary condition:

$$|v(x)| \to 0 \quad \text{uniformly} \quad \text{as} \quad |x| \to +\infty. \tag{1.3}$$

In addition to (1.3) one usually also assume following finite enstrophy condition.

$$\int_{\mathbb{R}^3} |\nabla v|^2 dx < \infty, \tag{1.4}$$

which is physically natural. It is well-known that any weak solution of (NS) satisfying (1.4) is smooth. Actually, the regularity result for the $L^\infty_t L^3_x$-weak solution of the non-stationary Navier-Stokes equations proved in [2] implies immediately that $v \in L^3(\mathbb{R}^3)$ is enough to guarantee the regularity. A long standing open question for solution of (NS) satisfying the conditions (1.3) and (1.4) is that if it is trivial (namely, $v = 0$ on $\mathbb{R}^3$), or not. We refer the book by Galdi([3]) for the details on the motivations and historical backgrounds on the problem and the related results. As a partial progress to the problem we mention that the condition $v \in L^{\frac{9}{2}}(\mathbb{R}^3)$ implies that $v = 0$ (see Theorem X.9.5, pp. 729 [3]). Another condition, $\Delta v \in L^{\frac{6}{5}}(\mathbb{R}^3)$ is also shown to imply $v = 0$([1]). For studies on the Liouville type problem in the *non-stationary* Navier-Stokes equations, we refer [4]. Our aim in this paper is to prove the following:

Theorem 1.1 *Let v be a smooth solution to (NS) on $\mathbb{R}^3$ satisfying (1.3). Suppose there exists $q \in [\frac{3}{2}, 3)$ such that $\omega \in L^q(\mathbb{R}^3)$. We set*

$$\Phi(x, y) := \frac{1}{4\pi} \frac{\omega(x) \cdot (x - y) \times (v(y) \times \omega(y))}{|x - y|^3} \tag{1.5}$$

for all $(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3$ with $x \neq y$, and define

$$\Phi_+(x, y) := \max\{0, \Phi(x, y)\}, \quad \Phi_-(x, y) := \max\{0, -\Phi(x, y)\}.$$

Then, either

$$v = 0 \quad \text{on} \quad \mathbb{R}^3, \tag{1.6}$$

or

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \Phi_+(x, y) dx dy = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \Phi_-(x, y) dx dy = +\infty. \tag{1.7}$$

Remark 1.1 One can show that if $\omega \in L^{\frac{9}{5}}(\mathbb{R}^3)$ is satisfied together with (1.3), then (1.6) holds. In order to see this we first recall the estimate of the Riesz potential on $\mathbb{R}^3$([5]),

$$\|I_\alpha(f)\|_{L^q} \le C\|f\|_{L^p}, \quad \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{3}, \quad 1 \le p < q < \infty, \tag{1.8}$$

where

$$I_\alpha(f) := C\int_{\mathbb{R}^3} \frac{f(y)}{|x-y|^{3-\alpha}} dy, \quad 0 < \alpha < 3$$

for a positive constant $C = C(\alpha)$. Applying (1.8) with $\alpha = 1$, we obtain by the Hölder inequality,

$$\begin{aligned}
\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |\Phi(x,y)| dy dx &\le \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|\omega(x)||\omega(y)||v(y)|}{|x-y|^2} dy dx \\
&\le \left(\int_{\mathbb{R}^3} |\omega(x)|^{\frac{9}{5}} dx\right)^{\frac{5}{9}} \left\{\int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} \frac{|\omega(y)||v(y)|}{|x-y|^2} dy\right)^{\frac{9}{4}} dx\right\}^{\frac{4}{9}} \\
&\le C\|\omega\|_{L^{\frac{9}{5}}} \left(\int_{\mathbb{R}^3} |\omega|^{\frac{9}{7}} |v|^{\frac{9}{7}} dx\right)^{\frac{7}{9}} \\
&\le C\|\omega\|_{L^{\frac{9}{5}}} \left(\int_{\mathbb{R}^3} |\omega|^{\frac{9}{5}} dx\right)^{\frac{5}{9}} \left(\int_{\mathbb{R}^3} |v|^{\frac{9}{2}} dx\right)^{\frac{2}{9}} \\
&\le C\|\omega\|^2_{L^{\frac{9}{5}}} \|\nabla v\|_{L^{\frac{9}{5}}} \le C\|\omega\|^3_{L^{\frac{9}{5}}} < +\infty,
\end{aligned}$$

where we used the Sobolev and the Calderon-Zygmund inequalities

$$\|v\|_{L^{\frac{9}{2}}} \le C\|\nabla v\|_{L^{\frac{9}{5}}} \le C\|\omega\|_{L^{\frac{9}{5}}} \tag{1.9}$$

in the last step. Thus, by the Fubini-Tonelli theorem, (1.7) cannot hold, and we are lead to (1.6) by application of the above theorem. We note that by (1.9) the condition $\omega \in L^{\frac{9}{5}}(\mathbb{R}^3)$, on the other hand, implies the previously known sufficient condition $v \in L^{\frac{9}{2}}(\mathbb{R}^3)$ of [3] mentioned above.

2 Proof of the main theorem

We first establish integrability conditions on the vector fields for the Biot-Savart's formula in $\mathbb{R}^3$.

Proposition 2.1 *Let $\xi = \xi(x) = (\xi_1(x), \xi_2(x), \xi_3(x))$ and $\eta = \eta(x) = (\eta_1(x), \eta_2(x), \eta_3(x))$ be smooth vector fields on $\mathbb{R}^3$. Suppose there exists $q \in [1, 3)$ such that $\eta \in L^q(\mathbb{R}^3)$. Let ξ solve*

$$\Delta \xi = -\nabla \times \eta, \tag{2.1}$$

under the boundary condition; either

$$|\xi(x)| \to 0 \quad \textit{uniformly} \quad \textit{as} \quad |x| \to +\infty, \tag{2.2}$$

or

$$\xi \in L^s(\mathbb{R}^3) \quad \textit{for some} \quad s \in [1, \infty). \tag{2.3}$$

Then, the solution of (2.1) is given by

$$\xi(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x-y) \times \eta(y)}{|x-y|^3} dy \quad \forall x \in \mathbb{R}^3. \tag{2.4}$$

Proof We introduce a cut-off function $\sigma \in C_0^\infty(\mathbb{R}^N)$ such that

$$\sigma(|x|) = \begin{cases} 1 & \text{if } |x| < 1, \\ 0 & \text{if } |x| > 2, \end{cases}$$

and $0 \le \sigma(x) \le 1$ for $1 < |x| < 2$. For each $R > 0$ we define $\sigma_R(x) := \sigma\left(\frac{|x|}{R}\right)$. Given $\varepsilon > 0$ we denote $B_\varepsilon(y) = \{x \in \mathbb{R}^3 \,|\, |x-y| < \varepsilon\}$. Let us fix $y \in \mathbb{R}^3$ and $\varepsilon \in (0, \frac{R}{2})$. We multiply (2.1) by $\frac{\sigma_R(|x-y|)}{|x-y|}$, and integrate it with respect to the variable x over $\mathbb{R}^3 \setminus B_\varepsilon(y)$. Then,

$$\int_{\{|x-y|>\varepsilon\}} \frac{\Delta \xi \, \sigma_R}{|x-y|} dx = -\int_{\{|x-y|>\varepsilon\}} \frac{\sigma_R \nabla \times \eta(y)}{|x-y|} dx. \tag{2.5}$$

Since $\Delta \frac{1}{|x-y|} = 0$ on $\mathbb{R}^3 \setminus B_\varepsilon(y)$, one has

$$\begin{aligned} \frac{\Delta \xi \sigma_R}{|x-y|} &= \sum_{i=1}^3 \partial_{x_i} \left(\frac{\partial_{x_i} \xi \sigma_R}{|x-y|} \right) - \sum_{i=1}^3 \partial_{x_i} \left(\frac{\xi \partial_{x_i} \sigma_R}{|x-y|} \right) \\ &\quad - \sum_{i=1}^3 \partial_{x_i} \left(\xi \sigma_R \partial_{x_i} (\frac{1}{|x-y|}) \right) + \frac{\xi \Delta \sigma_R}{|x-y|} + 2 \sum_{i=1}^3 \xi \partial_{x_i} (\frac{1}{|x-y|}) \partial_{x_i} \sigma_R. \end{aligned}$$

Therefore, applying the divergence theorem, and observing $\partial_\nu \sigma_R = 0$ on $\partial B_\varepsilon(y)$, we have

$$\begin{aligned}\int_{\{|x-y|>\varepsilon\}} \frac{\Delta\xi\sigma_R}{|x-y|}dx = &\int_{\{|x-y|=\varepsilon\}} \frac{\partial_\nu \xi}{|x-y|}dS \\ &- \int_{\{|x-y|=\varepsilon\}} \frac{\xi}{|x-y|^2}dS + \int_{\{|x-y|>\varepsilon\}} \frac{\xi\Delta\sigma_R}{|x-y|}dx \\ &-2\int_{\{|x-y|>\varepsilon\}} \frac{(x-y)\cdot\nabla\sigma_R\,\xi}{|x-y|^3}dx \end{aligned} \tag{2.6}$$

where $\partial_\nu(\cdot)$ denotes the outward normal derivative on $\partial B_\varepsilon(y)$. Passing $\varepsilon \to 0$, one can easily compute that

$$\begin{aligned}\text{RHS of (2.6)} \quad &\to \quad -4\pi\xi(y) + \int_{\mathbb{R}^3} \frac{\xi\Delta\sigma_R}{|x-y|}dx - 2\int_{\mathbb{R}^3} \frac{(x-y)\cdot\nabla\sigma_R\,\xi}{|x-y|^3}dx \\ &:= \quad I_1 + I_2 + I_3. \end{aligned} \tag{2.7}$$

Next, using the formula

$$\frac{\sigma_R \nabla\times\eta}{|x-y|} = \nabla\times\left(\frac{\sigma_R\eta}{|x-y|}\right) - \frac{\nabla\sigma_R\times\eta}{|x-y|} + \frac{(x-y)\times\eta\sigma_R}{|x-y|^3},$$

and using the divergence theorem, we obtain the following representation for the right hand side of (2.5).

$$\begin{aligned}\int_{\{|x-y|>\varepsilon\}} \frac{\sigma_R\nabla\times\eta}{|x-y|}dx = &\int_{\{|x-y|=\varepsilon\}} \nu\times\left(\frac{\eta}{|x-y|}\right)dS \\ &- \int_{\{|x-y|>\varepsilon\}} \frac{\nabla\sigma_R\times\eta}{|x-y|}dx + \int_{\{|x-y|>\varepsilon\}} \frac{(x-y)\times\eta\sigma_R}{|x-y|^3}dx, \end{aligned} \tag{2.8}$$

where we denoted $\nu = \frac{y-x}{|y-x|}$, the outward unit normal vector on $\partial B_\varepsilon(y)$. Passing $\varepsilon \to 0$, we easily deduce

$$\begin{aligned}\text{RHS of (2.8)} \quad &\to \quad -\int_{\mathbb{R}^3} \frac{\nabla\sigma_R\times\eta}{|x-y|}dx + \int_{\mathbb{R}^3} \frac{(x-y)\times\eta\sigma_R}{|x-y|^3}dx \\ &:= \quad J_1 + J_2 \quad \text{as} \quad \varepsilon\to 0. \end{aligned} \tag{2.9}$$

We now pass $R \to \infty$ for each term of (2.7) and (2.9) respectively below. Under the boundary condition (2.2) we estimate:

$$
\begin{aligned}
|I_2| &\le \int_{\{R \le |x-y| \le 2R\}} \frac{|\xi(x)||\Delta\sigma_R(x-y)|}{|x-y|} dx \\
&\le \frac{\|\Delta\sigma\|_{L^\infty}}{R^2} \sup_{R \le |x| \le 2R} |\xi(x)| \left(\int_{\{R \le |x-y| \le 2R\}} dx\right)^{\frac{2}{3}} \left(\int_{\{R \le |x-y| \le 2R\}} \frac{dx}{|x-y|^3}\right)^{\frac{1}{3}} \\
&\le C\|\Delta\sigma\|_{L^\infty} \left(\int_R^{2R} \frac{dr}{r}\right)^{\frac{2}{3}} \sup_{R \le |x-y| \le 2R} |\xi(x)| \to 0
\end{aligned}
$$

as $R \to \infty$ by the assumption (2.2), while under the condition (2.3) we have

$$
\begin{aligned}
|I_2| &\le \int_{\{R \le |x-y| \le 2R\}} \frac{|\xi(x)||\Delta\sigma_R(x-y)|}{|x-y|} dx \\
&\le \frac{\|\Delta\sigma\|_{L^\infty}}{R^2} \|\xi\|_{L^s} \left(\int_{\{0 \le |x-y| \le 2R\}} \frac{dx}{|x-y|^{\frac{s}{s-1}}}\right)^{\frac{s-1}{s}} \\
&\le CR^{-\frac{3}{s}} \|\Delta\sigma\|_{L^\infty} \|\xi\|_{L^s} \to 0
\end{aligned}
$$

as $R \to \infty$. Similarly, under (2.2)

$$
\begin{aligned}
|I_3| &\le 2\int_{\{R \le |x-y| \le 2R\}} \frac{|\xi(x)||\nabla\sigma_R(x-y)|}{|x-y|^2} dx \\
&\le \frac{C\|\nabla\sigma\|_{L^\infty}}{R} \sup_{R \le |x| \le 2R} |\xi(x)| \left(\int_{\{R \le |x-y| \le 2R\}} dx\right)^{\frac{1}{3}} \left(\int_{\{R \le |x-y| \le 2R\}} \frac{dx}{|x-y|^3}\right)^{\frac{2}{3}} \\
&\le C\|\nabla\sigma\|_{L^\infty} \left(\int_R^{2R} \frac{dr}{r}\right)^{\frac{2}{3}} \sup_{R \le |x-y| \le 2R} |\xi(x)| \to 0
\end{aligned}
$$

as $R \to \infty$, while under the condition (2.3) we estimate

$$
\begin{aligned}
|I_3| &\le 2\int_{\{R \le |x-y| \le 2R\}} \frac{|\xi(x)||\nabla\sigma_R(x-y)|}{|x-y|^2} dx \\
&\le \frac{C\|\nabla\sigma\|_{L^\infty}}{R} \|\xi\|_{L^s(R \le |x-y| \le 2R)} \left(\int_{\{0 \le |x-y| \le 2R\}} \frac{dx}{|x-y|^{\frac{2s}{s-1}}}\right)^{\frac{s-1}{s}} \\
&\le CR^{-\frac{3}{s}} \|\nabla\sigma\|_{L^\infty} \|\xi\|_{L^s} \to 0
\end{aligned}
$$

as $R \to \infty$. Therefore, the right hand side of (2.6) converges to $-4\pi\xi(y)$ as $R \to \infty$. For J_1, J_2 we estimate

$$
\begin{aligned}
|J_1| &\leq \int_{\{R \leq |x-y| \leq 2R\}} \frac{|\nabla \sigma_R||\eta|}{|x-y|} dx \\
&\leq \frac{C\|\nabla \sigma\|_{L^\infty}}{R} \|\eta\|_{L^q(R \leq |x-y| \leq 2R)} \left(\int_{\{0 \leq |x-y| \leq 2R\}} \frac{dx}{|x-y|^{\frac{q}{q-1}}} \right)^{\frac{q-1}{q}} \\
&\leq C\|\nabla \sigma\|_{L^\infty} \|\eta\|_{L^q(R \leq |x-y| \leq 2R)} R^{-\frac{2}{q}} \to 0
\end{aligned}
$$

as $R \to \infty$. In pasing $R \to \infty$ in J_2 of (2.9), in order to use the dominated convergence theorem, we estimate

$$
\begin{aligned}
\int_{\mathbb{R}^3} \left| \frac{(x-y) \times \eta(y)}{|x-y|^3} \right| dx &\leq \int_{\{|x-y|<1\}} \frac{|\eta|}{|x-y|^2} dx + \int_{\{|x-y| \geq 1\}} \frac{|\eta|}{|x-y|^2} dx \\
&:= J_{21} + J_{22}. \qquad (2.10)
\end{aligned}
$$

J_{21} is easy to handle as follows.

$$
J_{21} \leq \|\eta\|_{L^\infty(B_1(y))} \int_{\{|x-y|<1\}} \frac{dx}{|x-y|^2} = 4\pi \|\eta\|_{L^\infty(B_1(y))} < +\infty. \qquad (2.11)
$$

For J_{22} we estimate

$$
\begin{aligned}
J_{22} &\leq \left(\int_{\mathbb{R}^3} |\eta|^q dx \right)^{\frac{1}{q}} \left(\int_{\{|x-y|>1\}} \frac{dx}{|x-y|^{\frac{2q}{q-1}}} \right)^{\frac{q-1}{q}} \\
&\leq C\|\eta\|_{L^q} \left(\int_1^\infty r^{\frac{-2}{q-1}} dr \right)^{\frac{q-1}{q}} < +\infty, \qquad (2.12)
\end{aligned}
$$

if $1 < q < 3$. In the case of $q = 1$ we estimate simply

$$
J_{22} \leq \int_{\{|x-y|>1\}} |\eta| dx \leq \|\eta\|_{L^1}. \qquad (2.13)
$$

Estimates of (2.10)-(2.13) imply

$$
\int_{\mathbb{R}^3} \left| \frac{(x-y) \times \eta(y)}{|x-y|^3} \right| dx < +\infty.
$$

Summarising the above computations, one can pass first $\varepsilon \to 0$, and then $R \to +\infty$ in (2.5), applying the dominated convergence theorem, to obtain finally (2.4). $\square$

Corollary 2.1 *Let v be a smooth solution to (1.1)-(1.3) such that $\omega \in L^q(\mathbb{R}^3)$ for some $q \in [\frac{3}{2}, 3)$. Then, we have*

$$v(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x-y) \times \omega(y)}{|x-y|^3} dy, \tag{2.14}$$

and

$$\omega(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x-y) \times (v(y) \times \omega(y))}{|x-y|^3} dy. \tag{2.15}$$

Proof Taking curl of the defining equation of the vorticity, $\nabla \times v = \omega$, using div $v = 0$, we have

$$\Delta v = -\nabla \times \omega,$$

which provides us with (2.14) immediately by application of Proposition 2.1. In order to show (2.15) we recall that, using the vector identity $\frac{1}{2}\nabla |v|^2 = (v \cdot \nabla)v + v \times (\nabla \times v)$, one can rewrite (1.1)-(1.2) as

$$-v \times \omega = -\nabla \left(p + \frac{1}{2}|v|^2 \right) + \Delta v.$$

Taking curl on this, we obtain

$$\Delta \omega = -\nabla \times (v \times \omega).$$

The formula (2.15) is deduced immediately from this equations by applying the proposition 2.1. For the allowed rage of q we recall the Sobolev and the Calderon-Zygmund inequalities([5]),

$$\|v\|_{L^{\frac{3q}{3-q}}} \leq C\|\nabla v\|_{L^q} \leq C\|\omega\|_{L^q}, \quad 1 < q < 3, \tag{2.16}$$

which imply $v \times \omega \in L^{\frac{3q}{6-q}}(\mathbb{R}^3)$ if $\omega \in L^q(\mathbb{R}^3)$. We also note that $\frac{3}{2} \leq q < 3$ if and only if $1 \leq \frac{3q}{6-q} < 3$. $\square$

Proof of Theorem 1.1 Under the hypothesis (1.3) and $\omega \in L^q(\mathbb{R}^3)$ with

$q \in [\frac{3}{2}, 3)$ both of the relations (2.14) and (2.15) are valid. We first prove the following.

Claim: For each $x, y \in \mathbb{R}^3$

$$0 \le |\omega(x)|^2 = \int_{\mathbb{R}^3} \Phi(x,y)dy \le \int_{\mathbb{R}^3} |\Phi(x,y)|dy < +\infty, \tag{2.17}$$

and

$$0 = \int_{\mathbb{R}^3} \Phi(x,y)dx \le \int_{\mathbb{R}^3} |\Phi(x,y)|dx < +\infty. \tag{2.18}$$

Proof of the claim: We verify the following:

$$\int_{\mathbb{R}^3} |\Phi(x,y)|dy + \int_{\mathbb{R}^3} |\Phi(x,y)|dx < \infty \quad \forall (x,y) \in \mathbb{R}^3 \times \mathbb{R}^3. \tag{2.19}$$

Decomposing the integral, and using the Höolder inequality, we estimate

$$\begin{aligned}
\int_{\mathbb{R}^3} |\Phi(x,y)|dy &\le |\omega(x)| \left(\int_{\{|x-y|\le 1\}} \frac{|v(y)||\omega(y)|}{|x-y|^2} dy + \int_{\{|x-y|>1\}} \frac{|v(y)||\omega(y)|}{|x-y|^2} dy \right) \\
&\le |\omega(x)| \|v\|_{L^\infty(B_1(x))} \|\omega\|_{L^\infty(B_1(x))} \int_{\{|x-y|\le 1\}} \frac{dy}{|x-y|^2} \\
&\quad + |\omega(x)| \|v\|_{L^{\frac{3q}{3-q}}} \|\omega\|_{L^q} \left(\int_{\{|x-y|\ge 1\}} \frac{dy}{|x-y|^{\frac{6q}{4q-6}}} \right)^{\frac{4q-6}{3q}} \\
&\le C|\omega(x)| \|v\|_{L^\infty(B_1(x))} \|\omega\|_{L^\infty(B_1(x))} \\
&\quad + C|\omega(x)| \|\omega\|_{L^q}^2 \left(\int_1^\infty r^{\frac{q-6}{2q-3}} dr \right)^{\frac{4q-6}{3q}} < +\infty,
\end{aligned} \tag{2.20}$$

where we used (2.16) and the fact that $\frac{q-6}{3q-3} < -1$ if $\frac{3}{2} < q < 3$. In the case $q = \frac{3}{2}$ we estimate, instead,

$$\begin{aligned}
\int_{\mathbb{R}^3} |\Phi(x,y)|dy &\le |\omega(x)| \left(\int_{\{|x-y|\le 1\}} \frac{|v(y)||\omega(y)|}{|x-y|^2} dy + \int_{\{|x-y|>1\}} \frac{|v(y)||\omega(y)|}{|x-y|^2} dy \right) \\
&\le |\omega(x)| \|v\|_{L^\infty(B_1(x))} \|\omega\|_{L^\infty(B_1(x))} + |\omega(x)| \|v\|_{L^3} \|\omega\|_{L^{\frac{3}{2}}} < +\infty.
\end{aligned} \tag{2.21}$$

We also have

$$\begin{aligned}\int_{\mathbb{R}^3}|\Phi(x,y)|dx &\le |v(y)||\omega(y)|\left(\int_{\{|x-y|\le 1\}}\frac{|\omega(x)|}{|x-y|^2}dx+\int_{\{|x-y|>1\}}\frac{|\omega(x)|}{|x-y|^2}dx\right)\\ &\le C|v(y)||\omega(y)|\|\omega\|_{L^\infty(B_1(y))}+|v(y)||\omega(y)|\|\omega\|_{L^q}\left(\int_{\{|x-y|>1\}}\frac{dx}{|x-y|^{\frac{2q}{q-1}}}\right)^{\frac{q-1}{q}}\\ &\le C|v(y)||\omega(y)|\|\omega\|_{L^\infty(B_1(y))}+C|v(y)||\omega(y)|\|\omega\|_{L^q}\left(\int_1^\infty r^{-\frac{2}{q-1}}dr\right)^{\frac{q-1}{q}}<+\infty,\end{aligned} \tag{2.22}$$

where we used the fact that $-\frac{2}{q-1}<-1$ if $\frac{3}{2}\le q<3$. From (2.15) we immediately obtain

$$\begin{aligned}\int_{\mathbb{R}^3}\Phi(x,y)dy &= \omega(x)\cdot\left(\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{(x-y)\times(v(y)\times\omega(y))}{|x-y|^3}dy\right)\\ &= |\omega(x)|^2\ge 0,\quad \forall x\in\mathbb{R}^3\end{aligned} \tag{2.23}$$

and combining this with (2.20), we deduce (2.17). On the other hand, using (2.14), we find

$$\begin{aligned}\int_{\mathbb{R}^3}\Phi(x,y)dx &= \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{\omega(x)\cdot(x-y)\times(v(y)\times\omega(y))}{|x-y|^3}dx\\ &= \left(\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{\omega(x)\times(x-y)}{|x-y|^3}dx\right)\cdot v(y)\times\omega(y)\\ &= v(y)\cdot v(y)\times\omega(y)=0\end{aligned} \tag{2.24}$$

for all $y\in\mathbb{R}^3$, and combining this with (2.22), we have proved (2.18). This completes the proof of the claim.

By the Fubini-Tonelli theorem we have

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\Phi_+(x,y)dxdy=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\Phi_+(x,y)dydx:=\mathcal{I}_+, \tag{2.25}$$

and

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\Phi_-(x,y)dxdy=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\Phi_-(x,y)dydx:=\mathcal{I}_-. \tag{2.26}$$

If (1.7) does not hold, then at least one of the two integrals $\mathcal{I}_+, \mathcal{I}_-$ is finite. In this case, using (2.25) and (2.26), we can interchange the order of integrations in repeated integral as follows.

$$\begin{aligned}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \Phi(x,y)dxdy &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \Phi_+(x,y)dxdy - \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \Phi_-(x,y)dxdy \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \Phi_+(x,y)dydx - \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \Phi_-(x,y)dydx \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \Phi(x,y)dydx. \qquad (2.27)\end{aligned}$$

Therefore, from (2.23) and (2.24) combined with (2.27) provide us with

$$\int_{\mathbb{R}^3} |\omega(x)|^2 dx = \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \Phi(x,y)dydx = \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \Phi(x,y)dxdy = 0.$$

Hence,

$$\omega = 0 \quad \text{on} \quad \mathbb{R}^3. \qquad (2.28)$$

We remark parenthetically that in deriving (2.28) it is not necessary to assume that $\int_{\mathbb{R}^3} |\omega(x)|^2 dx < +\infty$, and we do not need to restrict ourselves to $\omega \in L^2(\mathbb{R}^3)$. Hence, from (2.14) and (2.28), we we conclude $v = 0$ on $\mathbb{R}^3$. $\square$

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References

[1] D. Chae, *Liouville-type theorems for the forced Euler equations and the Navier-Stokes equations,* Comm. Math. Phy., **326**, (2014), pp. 37-48.

[2] L. Escauriaza, G. Seregin and V. Šverák, $L_{3,\infty}-$*solutions to the Navier-Stokes equations and backward uniqeness,* Russ. Math. Surveys, **58**, (2003), pp. 211-250.

[3] G. P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations: Steady-State Problems*, 2nd ed., Springer, (2011).

[4] G. Koch, N. Nadirashvili, G. Seregin and V. Šverák, *Liouville theorems for the Navier-Stokes equations and applications,* Acta Math., **203**, no. 1, (2009), pp.83–105.

[5] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions,* Princeton Univ. Press (1970).